

INTEGRAL CALCULUS ON QUANTUM EXTERIOR ALGEBRAS

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ABSTRACT. Hom-connections and associated integral forms have been introduced and studied by T.Brzeziński as an adjoint version of the usual notion of a connection in non-commutative geometry. Given a flat hom-connection on a differential calculus (Ω, d) over an algebra A yields the integral complex which for various algebras has been shown to be isomorphic to the de Rham complex. In this paper we shed further light on the question when the integral and the de Rham complex are isomorphic for an algebra A with a flat hom-connection. We specialise our study to the case where an n -dimensional differential calculus can be constructed on a quantum exterior algebra over an A -bimodule. Criteria are given for free bimodules with diagonal or upper triangular bimodule structure. Our results are illustrated for a differential calculus on a multivariate quantum polynomial algebra and for a differential calculus on Manin's quantum n -space.

1. INTRODUCTION

Let A be an algebra over a field K . A derivation $d : A \rightarrow \Omega^1$ of a K -algebra A into an A -bimodule is a K -linear map satisfying the Leibniz rule $d(ab) = ad(b) + d(a)b$ for all $a, b \in A$. The pair (Ω^1, d) is called a *first order differential calculus* (FODC) on A . More generally a differential graded algebra $\Omega = \bigoplus_{n \geq 0} \Omega^n$ is an \mathbb{N} -graded algebra with a linear mapping $d : \Omega \rightarrow \Omega$ of degree 1 that satisfies $d^2 = 0$ and the graded Leibniz rule. This means that $d(\Omega^n) \subseteq \Omega^{n+1}$, $d^2 = 0$ and for all homogeneous elements $a, b \in \Omega$ the graded Leibniz rule:

$$(1.1) \quad d(ab) = d(a)b + (-1)^{|a|}ad(b)$$

holds, where $|a|$ denotes the degree of a , i.e. $a \in \Omega^{|a|}$ (see for example [8]). We shall call (Ω, d) an n -dimensional differential calculus on A if $\Omega^m = 0$ for all $m \geq n$. The zero component $A = \Omega^0$ is a subring of Ω and hence Ω^n are A -bimodule for all $n > 0$. In particular $d : A \rightarrow \Omega^1$ is a bimodule derivation and (Ω^1, d) is a FODC over A . The elements of Ω^n are then called n -forms and the product of Ω is denoted by \wedge . Given an FODC (Ω^1, d) over A and a left A -module M , a map $\nabla : M \rightarrow \Omega^1 \otimes_A M$ satisfying

$$(1.2) \quad \nabla(am) = a\nabla(m) + d(a) \otimes_A m \quad \forall a \in A, m \in M$$

is called a connection in M . In [4] T.Brzeziński introduced an adjoint version of a connection by introducing what he had called hom-connection on an right A -module M : $\nabla : \text{Hom}_A(\Omega^1, M) \rightarrow M$ satisfying

$$(1.3) \quad \nabla(fa) = \nabla(f)a + f(d(a)) \quad \forall a \in A, f \in \text{Hom}_A(\Omega^1, M)$$

where $fa \in \text{Hom}_A(\Omega^1, M)$ is defined as $fa(\omega) = f(a\omega)$, for all $\omega \in \Omega^1$. In case the FODC (Ω^1, d) stems from a differential calculus (Ω, d) , then a hom-connection $\nabla = \nabla_0$ on M can be extended to maps $\nabla_m : \text{Hom}_A(\Omega^{m+1}, M) \rightarrow \text{Hom}_A(\Omega^m, M)$ with

$$(1.4) \quad \nabla_m(f)(v) = \nabla(fv) + (-1)^{m+1}f(dv), \quad \forall f \in \text{Hom}_A(\Omega^{m+1}, M), v \in \Omega^m.$$

If $\nabla_0 \nabla_1 = 0$, the hom-connection ∇_0 is called flat. In this paper we will be mostly interested in the case $M = A$. Set $\Omega_m^* := \text{Hom}_A(\Omega^m, A)$ as well as $\Omega^* = \bigoplus_m \Omega_m^*$ and define $\nabla : \Omega^* \rightarrow \Omega^*$ by $\nabla(f) = \nabla_m(f)$ for all $f \in \Omega_{m+1}^*$.

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If ∇_0 is flat, then (Ω^*, ∇) builds up the *integral complex*:

$$\cdots \xrightarrow{\nabla_3} \Omega_3^* \xrightarrow{\nabla_2} \Omega_2^* \xrightarrow{\nabla_1} \Omega_1^* \xrightarrow{\nabla_0} A$$

It had been shown in [5, 7] that for some finite dimensional differential calculi the integral complex is isomorphic to the *de Rham complex* given by (Ω, d) :

$$A \xrightarrow{d} \Omega_1 \xrightarrow{d} \Omega_2 \xrightarrow{d} \Omega_3 \xrightarrow{d} \cdots$$

i.e. for certain algebras A and n -dimensional differential calculi $\Omega = \bigoplus_{m=0}^n \Omega^m$ it had been proven that there exists an isomorphism of complexes of right A -modules, that is, the diagram

$$\begin{array}{ccccccc} \Omega_n^* & \xrightarrow{\nabla_{n-1}} & \Omega_{n-1}^* & \xrightarrow{\nabla_{n-2}} & \cdots & \xrightarrow{\nabla_1} & \Omega_1^* \xrightarrow{\nabla_0} A \\ \Theta_0 \uparrow & & \Theta_1 \uparrow & & & & \Theta_{n-1} \uparrow \quad \Theta \uparrow \\ A & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^{n-1} \xrightarrow{d} \Omega^n \end{array}$$

is commutative. Following T.Brzeziński [5], we say that A satisfies the *strong Poincaré duality* with respect to (Ω, d) and ∇ , if such an isomorphism between these two complexes exists. In this case there exist isomorphisms

$$(1.5) \quad \Theta_m : \Omega^m \simeq \Omega_{n-m}^*$$

for any $0 \leq m \leq n$ and which yields in particular an isomorphism of right A -modules:

$$(1.6) \quad \Theta : \Omega \simeq \text{Hom}_A(\Omega, A).$$

The purpose of this paper is to provide further examples of algebras whose corresponding de Rham and integral complexes are isomorphic with respect to some differential calculi which contributes to the general study of algebras with this property. The reader should be warned that the Poincaré duality in the sense of M.Van den Bergh [13] (see also the work of U.Krähmer [10]) is different.

2. TWISTED MULTI-DERIVATIONS AND HOM-CONNECTIONS

From Woronowicz' paper [14] it follows that any covariant differential calculi on a quantum group is determined by a certain family of maps which had been termed *twisted multi-derivations* in [7].

We recall from [7] that by a *right twisted multi-derivation* in an algebra A we mean a pair (∂, σ) , where $\sigma : A \rightarrow M_n(A)$ is an algebra homomorphism ($M_n(A)$ is the algebra of $n \times n$ matrices with entries from A) and $\partial : A \rightarrow A^n$ is a k -linear map such that, for all $a \in A$, $b \in B$,

$$(2.7) \quad \partial(ab) = \partial(a)\sigma(b) + a\partial(b).$$

Here A^n is understood as an $(A, M_n(A))$ -bimodule. We write $\sigma(a) = (\sigma_{ij}(a))_{i,j=1}^n$ and $\partial(a) = (\partial_i(a))_{i=1}^n$ for an element $a \in A$. Then (2.7) is equivalent to the following n equations

$$(2.8) \quad \partial_i(ab) = \sum_j \partial_j(a)\sigma_{ji}(b) + a\partial_i(b), \quad i = 1, 2, \dots, n.$$

Given a right twisted multi-derivation (∂, σ) on A we construct a FODC on the free left A -module

$$(2.9) \quad \Omega^1 = A^n = \bigoplus_{i=1}^n A\omega_i$$

with basis $\omega_1, \dots, \omega_n$ which becomes an A -bimodule by $\omega_i a = \sum_{j=1}^n \sigma_{ij}(a)\omega_j$ for all $1 \leq i \leq n$. The map

$$(2.10) \quad d : A \rightarrow \Omega^1, \quad a \mapsto \sum_{i=1}^n \partial_i(a)\omega_i$$

is a derivation and makes (Ω^1, d) a first order differential calculus on A .

A map $\sigma : A \rightarrow M_n(A)$ can be equivalently understood as an element of $M_n(\text{End}_k(A))$. Write

- for the product in $M_n(\text{End}_k(A))$, \mathbb{I} for the unit in $M_n(\text{End}_k(A))$ and σ^T for the transpose of σ .

Definition 2.1. Let (∂, σ) be a right twisted multi-derivation. We say that (∂, σ) is *free*, provided there exist algebra maps $\bar{\sigma} : A \rightarrow M_n(A)$ and $\hat{\sigma} : A \rightarrow M_n(A)$ such that

$$(2.11) \quad \bar{\sigma} \bullet \sigma^T = \mathbb{I}, \quad \sigma^T \bullet \bar{\sigma} = \mathbb{I},$$

$$(2.12) \quad \hat{\sigma} \bullet \bar{\sigma}^T = \mathbb{I}, \quad \bar{\sigma}^T \bullet \hat{\sigma} = \mathbb{I}.$$

Theorem [7, Theorem 3.4] showed that for any free right twisted multi-derivation $(\partial, \sigma; \bar{\sigma}, \hat{\sigma})$ on A , and associated first order differential calculus (Ω^1, d) with generators ω_i , the map

$$(2.13) \quad \nabla : \text{Hom}_A(\Omega^1, A) \rightarrow A, \quad f \mapsto \sum_i \partial_i^\sigma (f(\omega_i)).$$

is a hom-connection, where $\partial_i^\sigma := \sum_{j,k} \bar{\sigma}_{kj} \circ \partial_j \circ \hat{\sigma}_{ki}$, for each $i = 1, 2, \dots, n$. Moreover ∇ had been shown to be unique with respect to the property that $\nabla(\xi_i) = 0$, for all $i = 1, 2, \dots, n$, where $\xi_i : \Omega^1 \rightarrow A$ are right A -linear maps defined by $\xi_i(\omega_j) = \delta_{ij}$, $i, j = 1, 2, \dots, n$.

We shall be mostly interested in right twisted multi-derivation (∂, σ) that are upper triangular, for which $\sigma_{ij} = 0$ for all $i > j$ holds. It had been shown in [7, Proposition 3.3] that an upper triangular right twisted multi-derivation is free if and only if $\sigma_{11}, \dots, \sigma_{nn}$ are automorphisms of A .

3. DIFFERENTIAL CALCULI ON QUANTUM EXTERIOR ALGEBRAS

Let A be a unital associative algebra over a field K . Given an A -bimodule M which is free as left and right A -module with basis $\{\omega_1, \dots, \omega_n\}$ one defines the tensor algebra of M over A as

$$(3.14) \quad T_A(M) = A \oplus M \oplus (M \otimes M) \oplus M^{\otimes 3} \oplus \dots = \bigoplus_{n=0}^{\infty} M^{\otimes n}$$

which is a graded algebra whose product is the concatenation of tensors and whose zero component is A . Following [3, I.2.1] we call an $n \times n$ -matrix $Q = (q_{ij})$ over K a *multiplicatively antisymmetric matrix* if $q_{ij}q_{ji} = q_{ii} = 1$ for all i, j . The *quantum exterior algebra* of M over A with respect to a multiplicatively antisymmetric matrix Q is defined as

$$\bigwedge^Q(M) := T_A(M) / \langle \omega_i \otimes \omega_j + q_{ij} \omega_j \otimes \omega_i, \omega_i \otimes \omega_i \mid i, j = 1, \dots, n \rangle.$$

This construction for a vector space $M = V$ and a field $A = K$ appears in [11, 12]. The product of $\bigwedge^Q(M)$ is written as \wedge . The quantum exterior algebra is a free left and right A -module of rank 2^n with basis

$$\{1\} \cup \{\omega_{i_1} \wedge \omega_{i_2} \cdots \wedge \omega_{i_k} \mid i_1 < i_2 < \dots < i_k, 1 \leq k \leq n\}.$$

Write $\text{sup}(\omega_{i_1} \wedge \omega_{i_2} \cdots \wedge \omega_{i_k}) = \{i_1, i_2, \dots, i_k\}$ for any basis element. Given a bimodule derivation $d : A \rightarrow M$, we will examine when d can be extended to an exterior derivation of $\bigwedge^Q(M)$, i.e. to a graded map $d : \bigwedge^Q(M) \rightarrow \bigwedge^Q(M)$ of degree 1 such that $d^2 = 0$ and such that the graded Leibniz rule is satisfied.

Let us first examine the bimodule structure of M : A bimodule structure on a free left A -module M with basis $\{\omega_1, \dots, \omega_n\}$ is given by an algebra map $\sigma : A \rightarrow M_n(A)$ such that

$$(3.15) \quad \omega_i a = \sum_{j=1}^n \sigma_{ij}(a) \omega_j \quad \forall a \in A, i = 1, \dots, n.$$

We will also assume that $\omega_1, \dots, \omega_n$ is a basis of M as a right A -module, which is equivalent to assume the existence of a map $\bar{\sigma} : A \rightarrow M_n(A)$ such that

$$(3.16) \quad \bar{\sigma} \bullet \sigma^T = \mathbb{I}, \quad \sigma^T \bullet \bar{\sigma} = \mathbb{I}.$$

$$(3.17) \quad a \omega_i = \sum_{j=1}^n \omega_j \bar{\sigma}_{ji}(a) \quad \forall a \in A, i = 1, \dots, n.$$

A bimodule derivation $d : A \rightarrow M$ is given by maps $\partial_1, \dots, \partial_n : A \rightarrow A$ such that $d(a) = \sum_{i=1}^n \partial_i(a)\omega_i$ for all $a \in A$. To be compatible with the bimodule structure the equation $d(ab) = d(a)b + ad(b)$ translates into the n equations:

$$(3.18) \quad \partial_i(ab) = \sum_j \partial_j(a)\sigma_{ji}(b) + a\partial_i(b), \quad i = 1, 2, \dots, n.$$

Let $\partial(a) = (\partial_i(a))_{i=1}^n$ for any $a \in A$. Then the pair (∂, σ) had been termed a right twisted multi-derivation in [7]. The associated A -bimodule and derivation are denoted by (Ω^1, d) and called a first order differential calculus on A .

Proposition 3.1. *Let (∂, σ) be a right twisted multi-derivation of rank n on a K -algebra A with associated FODC (Ω^1, d) . Let Q be an $n \times n$ multiplicatively antisymmetric matrix over k . Then $d : A \rightarrow \Omega^1$ can be extended to make $\Omega = \bigwedge^Q(\Omega^1)$ an n -dimensional differential calculus on A with $d(\omega_i) = 0$ for all $i = 1, \dots, n$ if and only if*

$$(3.19) \quad \partial_i \partial_j = q_{ji} \partial_j \partial_i, \quad \text{and} \quad \partial_i \sigma_{kj} - q_{ji} \partial_j \sigma_{ki} = q_{ji} \sigma_{kj} \partial_i - \sigma_{ki} \partial_j \quad \forall i < j, \forall k.$$

Proof. Suppose d extends to make Ω a differential calculus on A with $d(\omega_i) = 0$. Then for all $a \in A$ and $k = 1, \dots, n$ the following equations hold:

$$(3.20) \quad d(\omega_k a) = d(\omega_k)a - \omega_k \wedge d(a) = \sum_{j=1}^n -\omega_k \wedge \partial_j(a)\omega_j = \sum_{i,j=1}^n -\sigma_{ki}(\partial_j(a))\omega_i \wedge \omega_j$$

$$(3.21) \quad d\left(\sum_{j=1}^n \sigma_{kj}(a)\omega_j\right) = \sum_{i,j=1}^n \partial_i(\sigma_{kj}(a))\omega_i \wedge \omega_j + \sum_{j=1}^n \sigma_{kj}(a)d(\omega_j) = \sum_{i,j=1}^n \partial_i(\sigma_{kj}(a))\omega_i \wedge \omega_j$$

Hence, as $\omega_k a = \sum_{j=1}^n \sigma_{kj}(a)\omega_j$ and $\omega_j \wedge \omega_i = -q_{ji}\omega_i \wedge \omega_j$ for $i < j$, we have

$$(3.22) \quad -\sigma_{ki} \partial_j + q_{ji} \sigma_{kj} \partial_i = \partial_i \sigma_{kj} - q_{ji} \partial_j \sigma_{ki} \quad \forall i < j$$

Furthermore $d^2 = 0$ implies for all $a \in A$:

$$(3.23) \quad 0 = d^2(a) = \sum_{i,j=1}^n \partial_i \partial_j(a)\omega_i \wedge \omega_j = \sum_{i < j} (\partial_i \partial_j - q_{ji} \partial_j \partial_i)(a)\omega_i \wedge \omega_j,$$

which shows $\partial_i \partial_j = q_{ji} \partial_j \partial_i$, for $i < j$.

On the other hand if (3.19) holds, then set for any homogeneous element $a\omega \in \Omega^m$ with $a \in A$ and $\omega = \omega_{j_1} \wedge \omega_{j_2} \wedge \dots \wedge \omega_{j_m}$, with $j_1 < j_2 < \dots < j_m$, a basis element of Ω^m :

$$(3.24) \quad d(a\omega) := d(a) \wedge \omega = \sum_{i=1}^n \partial_i(a)\omega_i \wedge \omega_{j_1} \wedge \omega_{j_2} \wedge \dots \wedge \omega_{j_m}.$$

We will show that $d : \Omega \rightarrow \Omega$ in that way, will satisfy $d^2 = 0$ and the graded Leibniz rule. For any $a\omega \in \Omega^m$ as above:

$$(3.25) \quad d^2(a\omega) = \sum_{i,j=1}^n \partial_i \partial_j(a)\omega_i \wedge \omega_j \wedge \omega = \sum_{i < j} (\partial_i \partial_j - q_{ji} \partial_j \partial_i)(a)\omega_i \wedge \omega_j \wedge \omega = 0$$

Since (3.18) implies that $\partial_i(1) = \sum_j \partial_j(1)\sigma_{ji}(1) + \partial_i(1) = 2\partial_i(1)$, as $\sigma_{ji}(1) = 0$ if $i \neq j$, we have $\partial_i(1) = 0$ and hence $d(\omega_i) = d(1) \wedge \omega_i = 0$ for all i .

We prove the graded Leibniz rule

$$(3.26) \quad d(a\omega \wedge b\nu) = d(a\omega) \wedge b\nu + (-1)^m a\omega \wedge d(b\nu)$$

inductively on the grade of ω , where $\omega = \omega_{j_1} \wedge \dots \wedge \omega_{j_m}$ and $\nu = \omega_{i_1} \wedge \dots \wedge \omega_{i_k}$ are basis elements of Ω and $a, b \in A$. For a $m = 0$, ie. $a\omega = a$, equation (3.26) follows from the definition and

$d(\nu) = 0$. Let $m > 0$ and suppose that (3.26) has been proven for all basis elements ω of grade $|\omega| \leq m-1$. Let ω be a basis element with $|\omega| = m$ and write $\omega = \omega' \wedge \omega_k$.

$$\begin{aligned}
 d(a\omega \wedge b\nu) &= d(a\omega' \wedge \omega_k \wedge b\nu) \\
 &= \sum_{j=1}^n d(a\omega' \wedge \sigma_{kj}(b)\omega_j \wedge \nu) \\
 &= \sum_{j=1}^n d(a\omega') \wedge \sigma_{kj}(b)\omega_j \wedge \nu + (-1)^{m-1} \sum_{j=1}^n a\omega' \wedge d(\sigma_{kj}(b)\omega_j \wedge \nu) \\
 &= d(a\omega') \wedge \omega_k \wedge b\nu + (-1)^{m-1} a\omega' \wedge \sum_{i,j=1}^n \partial_i(\sigma_{kj}(b))\omega_i \wedge \omega_j \wedge \nu \\
 &= d(a\omega) \wedge b\nu - (-1)^m a\omega' \wedge \sum_{i < j} [\partial_i(\sigma_{kj}(b)) - q_{ji}\partial_j(\sigma_{ki}(b))] \omega_i \wedge \omega_j \wedge \nu \\
 &= d(a\omega) \wedge b\nu + (-1)^m a\omega' \wedge \sum_{i < j} [\sigma_{ki}(\partial_j(b)) - q_{ji}\sigma_{kj}(\partial_i(b))] \omega_i \wedge \omega_j \wedge \nu \\
 &= d(a\omega) \wedge b\nu + (-1)^m a\omega' \wedge \sum_{i,j=1}^n \sigma_{ki}(\partial_j(b))\omega_i \wedge \omega_j \wedge \nu \\
 &= d(a\omega) \wedge b\nu + (-1)^m a\omega' \wedge \omega_k \wedge \sum_{j=1}^n \partial_j(b)\omega_j \wedge \nu \\
 &= d(a\omega) \wedge b\nu + (-1)^m a\omega \wedge d(b\nu)
 \end{aligned}$$

which shows the graded Leibniz rule, where the induction hypothesis has been used in the third line and where (3.19) has been used in the sixth line. \square

Suppose that (∂, σ) is a free right twisted multi-derivation satisfying the equations (3.19) and that (Ω, d) is the associated n -dimensional differential calculus over A for some $n \times n$ matrix Q . Then, as mentioned above, $\nabla : \text{Hom}_A(\Omega^1, A) \rightarrow A$ with $\nabla(f) = \sum_{i=1}^n \partial_i^\sigma(f(\omega_i))$ for all $f \in \text{Hom}_A(\Omega^1, A)$ is hom-connection. For each $1 \leq m < n$ one defines also $\nabla_m : \text{Hom}_A(\Omega^{m+1}, A) \rightarrow \text{Hom}_A(\Omega^m, A)$ with

$$(3.27) \quad \nabla_m(f)(u) = \nabla(fu) + (-1)^{m+1} f(d(u)), \quad \forall f \in \text{Hom}_A(\Omega^{m+1}, A), u \in \Omega^m,$$

where $fu \in \text{Hom}_A(\Omega^1, A)$ is defined by $fu(v) = f(u \wedge v)$ for all $v \in \Omega^1$. As every element $u \in \Omega^m$ can be uniquely written as a right A -linear combination of basis elements $\omega = \omega_{i_1} \wedge \cdots \wedge \omega_{i_m}$ and since $\nabla_m(f)$ is right A -linear and furthermore by Proposition 3.1 $d(\omega) = 0$ is satisfied, we conclude that for $u = \omega a$:

$$(3.28) \quad \nabla_m(f)(\omega a) = \nabla_m(f)(\omega)a = \nabla(f\omega)a + (-1)^{m+1} f(d(\omega))a = \nabla(f\omega)a$$

holds. If $\partial_i^\sigma(1) = 0$ for all i , the hom-connection is flat, because for any dual basis element $f = \beta_{s,t} \in \text{Hom}_A(\Omega^2, A)$ with $s < t$, i.e. $\beta_{s,t}(\omega_i \wedge \omega_j) = \delta_{s,i}\delta_{t,j}$ one has

$$\nabla(\nabla_1(f)) = \sum_{i=1}^n \partial_i^\sigma(\nabla_1(f)(\omega_i)) = \sum_{i=1}^n \partial_i^\sigma(\nabla(f\omega_i)) = \sum_{i=1}^n \sum_{j=1}^n \partial_i^\sigma(\partial_j^\sigma(f(\omega_i \wedge \omega_j))) = \partial_s^\sigma(\partial_t^\sigma(1)) = 0.$$

Set $\Omega^* = \text{Hom}_A(\Omega, A) = \bigoplus_{m=0}^n \text{Hom}_A(\Omega^m, A)$ and note that ∇ induces a map of degree -1 on Ω^* . We want to establish an isomorphism between the de Rham complex given by $d : \Omega \rightarrow \Omega$ and the integral complex given by $\nabla : \Omega^* \rightarrow \Omega^*$. More precisely we are looking for a bijective chain map $\Theta : (\Omega, d) \rightarrow (\Omega^*, \nabla)$ such that the following diagram commutes:

$$\begin{array}{ccccccc}
A & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} \cdots \xrightarrow{d} & \Omega^{n-1} & \xrightarrow{d} & \Omega^n \\
\Theta_0 \downarrow & & \Theta_1 \downarrow & & \Theta_{n-1} \downarrow & & \Theta_n \downarrow \\
\text{Hom}_A(\Omega^n, A) & \xrightarrow{\nabla_{n-1}} & \text{Hom}_A(\Omega^{n-1}, A) & \xrightarrow{\nabla_{n-2}} \cdots \xrightarrow{\nabla_1} & \text{Hom}_A(\Omega^1, A) & \xrightarrow{\nabla} & A
\end{array}$$

One attempt is to define the maps Θ_m via the dual basis element of Ω^n . Define

$$\bar{\omega} = \omega_1 \wedge \cdots \wedge \omega_n \in \Omega^n$$

for the base element of Ω^n . Let $\beta \in \Omega^{n*}$ be the dual basis of Ω^n as a right A -module, i.e. $\beta(\bar{\omega}a) = a$ for all $a \in A$. For any $0 \leq m < n$ define $\Theta_m : \Omega^m \rightarrow \text{Hom}_A(\Omega^{n-m}, A)$ through $\Theta_m(v) = (-1)^{m(n-1)}\beta v$ for all $v \in \Omega^m$. Note that $\Theta_n = \beta$. Moreover the maps Θ_m are right A -linear taking into account the right A -module structure of $\text{Hom}_A(\Omega^{n-m}, A)$, namely for $a \in A, v \in \Omega^m$ and $w \in \Omega^{n-m}$:

$$\Theta_m(va)(w) = (-1)^{m(n-1)}\beta(va \wedge w) = (-1)^{m(n-1)}\beta(v \wedge aw) = \Theta_m(v)(aw) = (\Theta_m(v)a)(w).$$

Hence $\Theta_m(va) = \Theta_m(v)a$.

For a certain class of twisted multi-derivations, extended to a quantum exterior algebra, we will show that the maps Θ_m are always isomorphisms. We say that a twisted multi-derivation (∂, σ) on an algebra A is upper triangular if $\sigma_{ij} = 0$ for all $i > j$. By [7, Proposition 3.3] any upper triangular twisted multi-derivation is free if and only if σ_{ii} are automorphisms of A for all i . The corresponding maps $\bar{\sigma}$ and $\hat{\sigma}$ are defined inductively by $\bar{\sigma}_{ii} = \sigma_{ii}^{-1}$ for all i , $\bar{\sigma}_{ij} = -\sum_{k=j}^{i-1} \sigma_{ii}^{-1} \sigma_{ki} \bar{\sigma}_{kj}$ for all $i > j$ and $\bar{\sigma}_{ij} = 0$ for $i < j$. The map $\hat{\sigma}$ is defined analogously using $\bar{\sigma}$.

Theorem 3.2. *Let (∂, σ) be a free upper triangular twisted multi-derivation on A with associated FODC (Ω^1, d) . Suppose that $d : A \rightarrow \Omega^1$ can be extended to an n -dimensional differential calculus (Ω, d) where $\Omega = \bigwedge^Q(\Omega^1)$ is the quantum exterior algebra of Ω^1 for some matrix Q . Then the following hold:*

- (1) $\bar{\omega}a = \det(\sigma)(a) \bar{\omega}$, for all $a \in A$, where $\det(\sigma) = \sigma_{11} \circ \cdots \circ \sigma_{nn}$.
- (2) The maps $\Theta_m : \Omega^m \rightarrow \text{Hom}_A(\Omega^{n-m}, A)$ given by $\Theta_m(v) = (-1)^{m(n-1)}\beta v$ for all $v \in \Omega^m$ are isomorphisms of right A -modules.
- (3) Moreover if

$$(3.29) \quad \partial_i^\sigma = \left(\prod_j q_{ij} \right) \det(\sigma)^{-1} \partial_i \det(\sigma) \quad \forall i = 1, \dots, n$$

holds, then $\Theta = (\Theta_m)_{m=0}^n$ is a chain map, that is, A satisfies the strong Poincaré duality with respect to (Ω, d) in the sense of T.Brzezinski.

Proof. (1) By the definition of the bimodule structure of $\bigwedge^Q(\Omega^1)$ and by the fact that $\bar{\sigma}$ is lower triangular we have

$$a\bar{\omega} = \sum_{j_n \geq n} \cdots \sum_{j_1 \geq 1} \omega_{j_1} \wedge \cdots \wedge \omega_{j_n} \bar{\sigma}_{nj_n} \circ \cdots \circ \bar{\sigma}_{1j_1}(a).$$

By the definition of the quantum exterior algebra the non-zero terms $\omega_{j_1} \wedge \cdots \wedge \omega_{j_n}$ must have distinct indices, i.e. $j_k \neq j_l$ for all $k \neq l$. In particular $j_n = n$ and hence inductively we can conclude that $j_i = i$ for all i . This shows that $a\bar{\omega} = \bar{\omega} \det(\sigma)^{-1}(a)$.

(2) For every basis element of $\omega = \omega_{i_1} \wedge \cdots \wedge \omega_{i_{n-m}}$ of Ω^{n-m} , there exists a unique complement basis element $\omega' = \omega'_{j_1} \wedge \cdots \wedge \omega'_{j_m}$ of Ω^m such that $\omega' \wedge \omega \neq 0$. Let C_ω be the non-zero scalar such that $\omega' \wedge \omega = C_\omega \bar{\omega}$. Let $f \in \text{Hom}_A(\Omega^{n-m}, A)$ be any non-zero element and set

$$a_\omega = (-1)^{m(n-1)} C_\omega^{-1} \det(\sigma)(f(\omega))$$

for any basis element $\omega \in \Omega^{n-m}$. Set $v = \sum a_\omega \omega'$. Then

$$\Theta_m(v)(\omega) = (-1)^{m(n-1)}\beta(a_\omega \omega' \wedge \omega) = (-1)^{m(n-1)}\beta(a_\omega C_\omega \bar{\omega}) = \det(\sigma)^{-1}(\det(\sigma)(f(\omega))) = f(\omega).$$

Hence $\Theta_m(v) = f$, which shows that Θ_m is surjective. To prove injectivity, assume that $v = \sum a_\omega \omega' \in \Omega^m$ is an element such that $\Theta_m(v)$ is the zero function. Then for any basis element $\omega \in \Omega^{n-m}$, one has

$$\Theta_m(v)(\omega) = (-1)^{m(n-1)} \beta(a_\omega \omega' \wedge \omega) = (-1)^{m(n-1)} C_\omega \det(\sigma)^{-1}(a_\omega) = 0$$

which implies a_ω to be zero. Thus $v = 0$ and Θ_m is an isomorphism.

(3) We will show that $(\Theta_m)_m$ is a chain map, i.e. that $\Theta_{m+1} \circ d = \nabla_{n-m-1} \circ \Theta_m$. Let $\omega = \omega_{j_1} \wedge \cdots \wedge \omega_{j_m}$ be a basis element of Ω^m and let $a \in A$. For any basis element $\nu = \omega_{k_1} \wedge \cdots \wedge \omega_{k_{n-m-1}} \in \Omega^{n-m-1}$ we have

$$\Theta_{m+1}(d(a\omega))(\nu) = (-1)^{(m+1)(n-1)} \sum_{i=1}^n \beta(\partial_i(a)\omega_i \wedge \omega \wedge \nu).$$

On the other hand

$$\nabla_{n-m-1}(\Theta_m(a\omega))(\nu) = (-1)^{m(n-1)} \nabla(\beta(a\omega \wedge \nu)) = (-1)^{m(n-1)} \sum_{i=1}^n \partial_i(\beta(a\omega \wedge \nu \wedge \omega_i)),$$

as $d(\nu) = 0$. Note that $\Theta_{m+1}(d(a\omega))(\nu) = 0$ and $\nabla_{n-m-1}(\Theta_m(a\omega))(\nu) = 0$ if $\text{sup}(\omega) \cap \text{sup}(\nu) \neq \emptyset$. Hence suppose that ω and ν have disjoint support. Then there exists a unique index i that does not belong to $\text{sup}(\omega) \cup \text{sup}(\nu)$. Let C be the constant such that

$$\omega \wedge \nu \wedge \omega_i = C\bar{\omega}.$$

Recall also that by the definition of the quantum exterior algebra we have:

$$\omega_i \wedge \omega \wedge \nu = \left(\prod_{j \neq i} -q_{ij} \right) \omega \wedge \nu \wedge \omega_i = (-1)^{n-1} C \left(\prod_j q_{ij} \right) \bar{\omega}.$$

Note that hypothesis (3.29) is moreover equivalent to

$$(3.30) \quad \partial_i^\sigma \circ \det(\sigma)^{-1} = \left(\prod_j q_{ij} \right) \det(\sigma)^{-1} \circ \partial_i$$

These equations yield now the following:

$$\begin{aligned} \Theta_{m+1}(d(a\omega))(\nu) &= (-1)^{(m+1)(n-1)} \beta(\partial_i(a)\omega_i \wedge \omega \wedge \nu) \\ &= (-1)^{m(n-1)} C \left(\prod_j q_{ij} \right) \beta(\partial_i(a)\bar{\omega}) \\ &= (-1)^{m(n-1)} C \left(\prod_j q_{ij} \right) \det(\sigma)^{-1}(\partial_i(a)) \\ &= (-1)^{m(n-1)} C \partial_i^\sigma \left(\det(\sigma)^{-1}(a) \right) \\ &= \partial_i^\sigma \left((-1)^{m(n-1)} C \beta(a\bar{\omega}) \right) \\ &= \partial_i^\sigma \left((-1)^{m(n-1)} \beta(a\omega \wedge \nu \wedge \omega_i) \right) = \nabla_{n-m-1}(\Theta_m(a\omega))(\nu) \end{aligned}$$

Thus $\Theta_{m+1} \circ d = \nabla_{n-m-1} \circ \Theta_m$. Hence Θ is a chain map between the de Rham and the integral complexes of right A -modules. \square

Remark 3.3. Let (∂, σ) be an upper-triangular twisted multi-derivation of rank n on A and let Q be an $n \times n$ matrix with $q_{ij}q_{ji} = q_{ii} = 1$. The conditions to extend the multi-derivations to the quantum exterior algebra $\Omega = \bigwedge^Q(\Omega^1)$ such that the complex of integral forms on A and the de Rham complex are isomorphic with respect to (Ω, d) are:

- (1) σ_{ii} is an automorphism of A for all i ;

- (2) $\partial_i \partial_j = q_{ji} \partial_j \partial_i$ for all $i < j$;
- (3) $\partial_i \sigma_{kj} - q_{ji} \sigma_{kj} \partial_i = q_{ji} \partial_j \sigma_{ki} - \sigma_{ki} \partial_j$ for all $i < j$ and all k ;
- (4) $\partial_i^\sigma = \left(\prod_j q_{ij} \right) \det(\sigma)^{-1} \partial_i \det(\sigma)$ for all i .

4. DIFFERENTIAL CALCULI FROM SKEW DERIVATIONS

The simplest bimodule structure on $\Omega^1 = A^n$ is a *diagonal* one, i.e. if $\sigma_{ij} = \delta_{ij} \sigma_i$ for all i, j where $\sigma_1, \dots, \sigma_n$ are endomorphisms of A . Moreover if σ is diagonal and (∂, σ) is a right twisted multi-derivation on A , then the maps ∂_i are right σ_i -derivations, i.e. for all $a, b \in A$ and i :

$$(4.31) \quad \partial_i(ab) = \partial_i(a) \sigma_i(b) + a \partial_i(b).$$

Conversely, given any right σ_i -derivations ∂_i on A , for $i = 1, \dots, n$ one can form a corresponding diagonal twisted multi-derivation (∂, σ) on A . Such *diagonal* twisted multi-derivation (∂, σ) is free if and only if the maps $\sigma_1, \dots, \sigma_n$ are automorphisms. The associated A -bimodule structure on $\Omega^1 = A^n$ with left A -basis $\omega_1, \dots, \omega_n$ is given by $\omega_i a = \sigma_i(a) \omega_i$ for all i and $a \in A$. From Proposition 3.1 we obtain the following corollary for diagonal bimodule structures.

Corollary 4.1. *Let A be an algebra over a field K , σ_i automorphisms and ∂_i right σ_i -skew derivations on A , for $i = 1, \dots, n$ and let (Ω^1, d) be the associated first order differential calculus on A .*

- (1) *The derivation $d : A \rightarrow \Omega^1$ extends to an n -dimensional differential calculus (Ω, d) where $\Omega = \bigwedge^Q(\Omega^1)$ is the quantum exterior algebra with respect to some Q such that $d(\omega_i) = 0$ for all $i = 1, \dots, n$ if and only if*

$$(4.32) \quad \partial_i \sigma_j = q_{ji} \sigma_j \partial_i \quad \text{and} \quad \partial_i \partial_j = q_{ji} \partial_j \partial_i \quad \forall i < j$$

- (2) *If $\partial_i \sigma_j = q_{ji} \sigma_j \partial_i$ for all i, j and $\partial_i \partial_j = q_{ji} \partial_j \partial_i$ for all $i < j$, then the de Rham and the integral complexes on A are isomorphic relative to (Ω, d) .*

Proof. (1) Since $\sigma_{ki} = 0$ for all $k \neq i$, equation (3.19) reduces to equation (4.32).

(2) Note that $\partial_i^\sigma = \sigma_i^{-1} \partial_i \sigma_i = \partial_i$. On the other hand by hypothesis $\partial_i \det(\sigma) = \left(\prod_j q_{ji} \right) \det(\sigma) \partial_i$. Hence

$$\left(\prod_j q_{ij} \right) \det(\sigma)^{-1} \partial_i \det(\sigma) = \partial_i = \partial_i^\sigma.$$

Thus by Theorem 3.2, A satisfies the strong Poincaré duality with respect to (Ω, d) in the sense of T.Brzeziński. \square

5. MULTIVARIATE QUANTUM POLYNOMIALS

Let K be a field, $n > 1$, and $Q = (q_{ij})$ a $n \times n$ multiplicatively antisymmetric matrix over K . The multivariate quantum polynomial algebra with respect to Q is defined as:

$$A = \mathcal{O}_Q(K^n) := K \langle x_1, \dots, x_n \rangle / \langle x_i x_j - q_{ij} x_j x_i \mid 1 \leq i, j \leq n \rangle.$$

This means that x_i and x_j commute up to the scalar q_{ij} in A . Moreover every element is a linear combination of ordered monomials $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ with $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. The set of n -tuples \mathbb{N}^n is a submonoid of \mathbb{Z}^n by componentwise addition. For any $\alpha \in \mathbb{Z}^n$ we set $x^\alpha = 0$ if there exists $i = 1, \dots, n$ such that $\alpha_i < 0$. Furthermore \mathbb{N}^n is partially ordered as follows: $\alpha \leq \beta$ if and only if $\alpha_i \leq \beta_i, i = 1, \dots, n$ for $\alpha, \beta \in \mathbb{N}^n$. If $\alpha \leq \beta$, then $\beta - \alpha \in \mathbb{N}^n$ and $x^{\beta-\alpha} \neq 0$.

For two generic monomials x^α and x^β with $\alpha, \beta \in \mathbb{N}^n$ one has

$$(5.33) \quad x^\alpha x^\beta = \left(\prod_{1 \leq j < i \leq n} q_{ij}^{\alpha_i \beta_j} \right) x^{\alpha+\beta} = \mu(\alpha, \beta) x^{\alpha+\beta},$$

where $\mu(\alpha, \beta) = \prod_{1 \leq j < i \leq n} q_{ij}^{\alpha_i \beta_j}$. The algebra A has been well-studied by Artamonov [1, 2] as well by Goodearl and Brown [3] and others. The Manin's quantum n -space is obtained in case there exists $q \in K$ with $q_{ij} = q$ for all $i < j$. In particular for $n = 2$ one obtains the quantum plane.

We define automorphisms $\sigma_1, \dots, \sigma_n$ and right σ_i -derivations of A as follows: For a generic monomial x^α with $\alpha \in \mathbb{N}^n$ one sets

$$(5.34) \quad \sigma_i(x^\alpha) := \lambda_i(\alpha)x^\alpha \quad \text{and} \quad \partial_i(x^\alpha) := \alpha_i \delta_i(\alpha)x^{\alpha - \epsilon^i}$$

where $\lambda_i(\alpha) = \prod_{j=1}^n q_{ij}^{\alpha_j}$, $\delta_i(\alpha) = \prod_{i < j} q_{ij}^{\alpha_j}$ and $\epsilon^i \in \mathbb{N}^n$ such that $\epsilon_j^i = \delta_{ij}$. Let $\bar{\delta}_i(\alpha) = \prod_{i > j} q_{ij}^{\alpha_j}$ and note that $\lambda_i(\alpha) = \delta_i(\alpha)\bar{\delta}_i(\alpha)$. Since $\mu(\alpha, \beta) = \mu(\alpha - \epsilon^i, \beta)\bar{\delta}_i(\beta)$ if $\alpha_i \neq 0$ and $\mu(\alpha, \beta) = \mu(\alpha, \beta - \epsilon^i)\delta_i(\alpha)^{-1}$ if $\beta_i \neq 0$, we have:

$$\begin{aligned} \partial_i(x^\alpha x^\beta) &= (\alpha_i + \beta_i)\mu(\alpha, \beta)\delta_i(\alpha + \beta)x^{\alpha + \beta - \epsilon^i} \\ &= \alpha_i \mu(\alpha - \epsilon^i, \beta)\bar{\delta}_i(\beta)\delta_i(\alpha)\delta_i(\beta)x^{\alpha - \epsilon^i + \beta} + \beta_i \mu(\alpha, \beta - \epsilon^i)\delta_i(\alpha)^{-1}\delta_i(\alpha)\delta_i(\beta)x^{\alpha + \beta - \epsilon^i} \\ &= \alpha_i \delta_i(\alpha)x^{\alpha - \epsilon^i} \lambda_i(\beta)x^\beta + x^\alpha \beta_i \delta_i(\beta)x^{\beta - \epsilon^i} \\ &= \partial_i(x^\alpha)\sigma_i(x^\beta) + x^\alpha \partial_i(x^\beta) \end{aligned}$$

Let $i < j$ and $\alpha \in \mathbb{N}^n$. Then $\delta_j(\alpha - \epsilon^i) = \delta_j(\alpha)$, while $\delta_i(\alpha - \epsilon^j) = \delta_i(\alpha)q_{ji}$. Hence

$$(5.35) \quad \partial_j(\partial_i(x^\alpha)) = \alpha_i \alpha_j \delta_i(\alpha)\delta_j(\alpha - \epsilon^i)x^{\alpha - \epsilon^i - \epsilon^j} = \alpha_i \alpha_j q_{ij} \delta_i(\alpha - \epsilon^j)\delta_j(\alpha)x^{\alpha - \epsilon^i - \epsilon^j} = q_{ij} \partial_i(\partial_j(x^\alpha))$$

Thus $\partial_j \partial_i = q_{ij} \partial_i \partial_j$ for all $i < j$.

Let $i \leq j$ and $\alpha \in \mathbb{N}^n$. Then

$$(5.36) \quad \sigma_i(\partial_j(x^\alpha)) = \alpha_j \delta_j(\alpha)\lambda_i(\alpha - \epsilon^j)x^{\alpha - \epsilon^j} = \alpha_j \delta_j(\alpha)\lambda_i(\alpha)q_{ji}x^{\alpha - \epsilon^j} = q_{ji}\lambda_i(\alpha)\partial_j(x^\alpha) = q_{ji}\partial_j(\sigma_i(x^\alpha)).$$

Hence $\sigma_i \partial_j = q_{ji} \partial_j \sigma_i$ for all $i \leq j$. By Corollary 4.1 we can conclude:

Corollary 5.1. *Let $A = \mathcal{O}_Q(K^n)$ be the multivariate quantum polynomial algebra and let $\Omega = \bigwedge^Q(\Omega^1)$ be the associated quantum exterior algebra. Then the derivation $d : A \rightarrow \Omega^1$ with $d(x^\alpha) = \sum_{i=1}^n \partial_i(x^\alpha)\omega_i$ makes Ω into a differential calculus such that the de Rham complex and the integral complex are isomorphic.*

6. MANIN'S QUANTUM n -SPACE

In this section we will show that for a special case of the multivariate quantum polynomial algebra there exists a differential calculus whose bimodule structure is not diagonal, but upper triangular and nevertheless the de Rham complex and the integral complex are isomorphic.

Let $q \in K \setminus \{0\}$. For the matrix $Q = (q_{ij})$ with $q_{ij} = q$ and $q_{ji} = q^{-1}$ for all $i < j$ and $q_{ii} = 1$, the algebra $\mathcal{O}_Q(K^n)$ is called the *coordinate ring of quantum n -space* or *Manin's quantum n -space* and will be denoted by $A = K_q[x_1, \dots, x_n]$. We have the following defining relations of the algebra A

$$(6.37) \quad x_i x_j = q x_j x_i, \quad i < j.$$

Note that for $\alpha \in \mathbb{N}^n$ and $1 \leq i \leq n$ we have:

$$\lambda_i(\alpha)x^\alpha x_i = x^{\alpha + \epsilon^i} = \bar{\lambda}_i(\alpha)x_i x^\alpha,$$

where

$$\lambda_i(\alpha) = \prod_{i < j} q^{\alpha_j} \quad \text{and} \quad \bar{\lambda}_i(\alpha) = \prod_{j < i} q^{-\alpha_j}.$$

More generally

$$x^{\alpha + \beta} = \left(\prod_{j=1}^{n-1} \lambda_j(\alpha)^{\beta_j} \right) x^\alpha x^\beta = \prod_{1 \leq s < j \leq n} q^{\alpha_s \beta_j} x^\alpha x^\beta$$

Let $\mu(\alpha, \beta)$ be the scalar such that $x^\alpha x^\beta = \mu(\alpha, \beta)x^{\alpha + \beta}$.

We take the following two-parameter first order differential calculus Ω^1 (see [9, p.468] for the case $p = q^2$ and [7, Example 3.9] for the case $n = 2$), which is freely generated by $\{\omega_1, \dots, \omega_n\}$ over A subject to the relations

$$(6.38) \quad \omega_i x_j = q x_j \omega_i + (p - 1)x_i \omega_j, \quad i < j,$$

$$(6.39) \quad \omega_i x_i = p x_i \omega_i,$$

$$(6.40) \quad \omega_j x_i = p q^{-1} x_i \omega_j, \quad i < j,$$

There exists an algebra map $\sigma : A \rightarrow M_n(A)$ whose associated matrix of endomorphisms $\sigma = (\sigma_{ij})$ is upper triangular and such that $\omega_i x^\alpha = \sum_{i \leq j} \sigma_{ij}(x^\alpha) \omega_j$. The next lemma will characterize the algebra map σ . For any $\alpha \in \mathbb{N}^n$ and $i = 1, \dots, n$ set $\pi_i(\alpha) = \prod_{s < i} p^{\alpha_s}$.

Lemma 6.1. *For $\alpha \in \mathbb{N}^n$ the entries of the matrix $\sigma(x^\alpha)$ are as follows $\sigma_{ij}(x^\alpha) = 0$ for $i > j$ and*

$$\sigma_{ij}(x^\alpha) = \eta_{ij}(\alpha) x^{\alpha + \epsilon^i - \epsilon^j} \quad \text{where} \quad \eta_{ij}(\alpha) = \begin{cases} \pi_j(\alpha) \bar{\lambda}_i(\alpha) \lambda_j(\alpha) (p^{\alpha_j} - 1) & \text{for } i < j, \\ \pi_i(\alpha) \bar{\lambda}_i(\alpha) \lambda_i(\alpha) p^{\alpha_i} & \text{for } i = j \end{cases}$$

Proof. Fix a number i between 1 and n . We prove the relations for σ_{ij} by induction on the length of α , which by length we mean $|\alpha| = \alpha_1 + \dots + \alpha_n$. For $|\alpha| = 0$ the relation is clear, because $\alpha_j = 0$ for all j , i.e. $x^\alpha = 1$. Hence $\omega_i x^\alpha = \omega_i$, i.e. $\sigma_{ij}(x^\alpha) = \delta_{ij}$. Since $p^{\alpha_j} - 1 = 0$ for all j and $p^{\alpha_i} = 1$ the relation holds.

Now suppose that $m \geq 0$ and that the relations (6.1) hold for all $\alpha \in \mathbb{N}^n$ of length m . Let $\beta \in \mathbb{N}^n$ be an element of length $m + 1$ and let k be the largest index j such that $\beta_j \neq 0$. Set $\alpha = \beta - \epsilon^k$, i.e. $\beta = \alpha + \epsilon^k$. We have to discuss the three cases $k < i$, $k = i$ and $k > i$.

If $k < i$, then for all $i < j$, $\alpha_j = 0$, i.e. $\sigma_{ij}(x^\alpha) = 0$. Hence

$$\omega_i x^\beta = \omega_i x^\alpha x_k = \sigma_{ii}(x^\alpha) \omega_i x_k = p q^{-1} \sigma_{ii}(x^\alpha) x_k \omega_i = p \pi_i(\alpha) q^{-1} \bar{\lambda}_i(\alpha) x^\alpha x_k \omega_i = \pi_i(\beta) \bar{\lambda}_i(\beta) x^\beta \omega_i,$$

since $\lambda_i(\alpha) = p^{\alpha_i} = 1$, $\pi_i(\alpha + \epsilon^k) = p \pi_i(\alpha)$ and $\bar{\lambda}_i(\alpha + \epsilon^k) = q^{-1} \bar{\lambda}_i(\alpha)$ for any $k < i$ and $\alpha \in \mathbb{N}^n$. Thus $\sigma_{ii}(x^\beta) = \pi_i(\beta) \bar{\lambda}_i(\beta) \lambda_i(\beta) p^{\beta_i} x^\beta$.

If $k = i$, then again $\sigma_{ij}(x^\alpha) = 0$ for all $j > i$. Moreover $\lambda_j(\alpha) = 1$ for all $j > i$. Thus

$$\omega_i x^\beta = \sigma_{ii}(x^\alpha) \omega_i x_i = \sigma_{ii}(x^\alpha) p x_i \omega_i = \pi_i(\alpha) \bar{\lambda}_i(\alpha) p^{\alpha_i + 1} x^\alpha x_i \omega_i = \pi_i(\beta) \bar{\lambda}_i(\beta) p^{\beta_i} x^\beta \omega_i,$$

since $\alpha_s = \beta_s$ for all $s < i$, i.e. $\pi_i(\beta) = \pi_i(\alpha)$ and $\bar{\lambda}_i(\beta) = \bar{\lambda}_i(\alpha)$.

If $i < k$, then note that $\sigma_{ij}(x^\alpha) = 0$ for all $k < j$, because $p^{\alpha_j} = 1$. Thus

$$\begin{aligned} \omega_i x^\beta &= \sigma_{ii}(x^\alpha) \omega_i x_k + \sum_{i < j < k} \sigma_{ij}(x^\alpha) \omega_j x_k + \sigma_{ik}(x^\alpha) \omega_k x_k \\ &= \sigma_{ii}(x^\alpha) [q x_k \omega_i + (p - 1) x_i \omega_k] + \sum_{i < j < k} \sigma_{ij}(x^\alpha) [q x_k \omega_j + (p - 1) x_j \omega_k] + \sigma_{ik}(x^\alpha) p x_k \omega_k \\ &= q \sigma_{ii}(x^\alpha) x_k \omega_i + \sum_{i < j < k} q \sigma_{ij}(x^\alpha) x_k \omega_j + \underbrace{\left[(p - 1) \sigma_{ii}(x^\alpha) x_i + \sum_{i < j < k} (p - 1) \sigma_{ij}(x^\alpha) x_j + p \sigma_{ik}(x^\alpha) x_k \right]}_{(*)} \omega_k \end{aligned}$$

Note that for any $j < k$ we have $q \lambda_j(\alpha) = \lambda_j(\beta)$. Hence $q \sigma_{ij}(x^\alpha) x_k = \sigma_{ij}(x^\beta)$ for all $j < k$. It is left to show that the expression $(*)$ equals $\sigma_{ik}(x^\beta)$. Recall that $\lambda_l(\alpha) x^\alpha x_l = x^{\alpha + \epsilon^l}$. Hence

$\lambda_j(\alpha)x^{\alpha+\epsilon^i-\epsilon^j}x_j = x^{\alpha+\epsilon^i}$. Note also that $p^{\alpha_j}\pi_j(\alpha) = \pi_{j+1}(\alpha)$.

$$\begin{aligned}
 (*) &= (p-1)\bar{\lambda}_i(\alpha) \left[\pi_i(\alpha)\lambda_i(\alpha)p^{\alpha_i}x^\alpha x_i + \sum_{i < j < k} \pi_j(\alpha)\lambda_j(\alpha)(p^{\alpha_j}-1)x^{\alpha+\epsilon^i-\epsilon^j}x_j \right] + p\sigma_{ik}(x^\alpha)x_k \\
 &= (p-1)\bar{\lambda}_i(\alpha) \left[p^{\alpha_i}\pi_i(\alpha) + \sum_{i < j < k} \pi_j(\alpha)(p^{\alpha_j}-1) \right] x^{\alpha+\epsilon^i} + p\sigma_{ik}(x^\alpha)x_k \\
 &= (p-1)\bar{\lambda}_i(\alpha) \left[\pi_{i+1}(\alpha) + \sum_{i < j < k} (\pi_{j+1}(\alpha) - \pi_j(\alpha)) \right] x^{\alpha+\epsilon^i} + p\pi_k(\alpha)\bar{\lambda}_i(\alpha)(p^{\alpha_k}-1)x^{\alpha+\epsilon^i} \\
 &= (p-1)\bar{\lambda}_i(\alpha) [\pi_{i+1}(\alpha) + \pi_k(\alpha) - \pi_{i+1}(\alpha)] x^{\alpha+\epsilon^i} + p\pi_k(\alpha)\bar{\lambda}_i(\alpha)(p^{\alpha_k}-1)x^{\alpha+\epsilon^i} \\
 &= \bar{\lambda}_i(\alpha) [(p-1)\pi_k(\alpha) + p\pi_k(\alpha)(p^{\alpha_k}-1)] x^{\alpha+\epsilon^i} \\
 &= \bar{\lambda}_i(\alpha)(p^{\alpha_k+1}-1)\pi_k(\alpha)x^{\alpha+\epsilon^i} \\
 &= \pi_k(\beta)\bar{\lambda}_i(\beta)\lambda_k(\beta)(p^{\beta_k}-1)x^{\beta+\epsilon^i-\epsilon^k} = \sigma_{ik}(x^\beta),
 \end{aligned}$$

since $\lambda_k(\beta) = 1 = \lambda_k(\alpha)$ and $\pi_k(\alpha) = \pi_k(\beta)$ as α and β differ only in the k th position. \square

We will define a derivation $d : K_q[x_1, \dots, x_n] \rightarrow \Omega^1$ such that $d(x_i) = \omega_i$ for all i . For any $\alpha \in \mathbb{N}^n$ we set $d(x^\alpha) = \sum_{i=1}^n \partial_i(x^\alpha)\omega_i$ where

$$(6.41) \quad \partial_i(x^\alpha) = \delta_i(\alpha)x^{\alpha-\epsilon^i} \quad \text{and} \quad \delta_i(\alpha) = \pi_i(\alpha)\lambda_i(\alpha)\frac{p^{\alpha_i}-1}{p-1}.$$

for all $i = 1, \dots, n$. Note that for i, k we have:

$$\delta_i(\alpha) = q^{\mp 1}\delta_i(\alpha \pm \epsilon^k), \quad \text{if } i < k \quad \text{and} \quad \delta_i(\alpha) = p^{\mp 1}\delta_i(\alpha \pm \epsilon^k), \quad \text{if } i > k.$$

Lemma 6.2. *The pair (∂, σ) is a right twisted multi-derivation of $K_q[x_1, \dots, x_n]$ satisfying the equations (3.19) with respect to the multiplicatively antisymmetric matrix Q' whose entries are $Q'_{ij} = p^{-1}q$ for $i < j$. In particular*

$$(6.42) \quad \partial_i\partial_j = pq^{-1}\partial_j\partial_i, \quad \forall i < j$$

holds as well as for all i, k, j :

$$\begin{aligned}
 \partial_i\sigma_{kj} &= pq^{-1}\sigma_{kj}\partial_i, & i < k \leq j \\
 \partial_i\sigma_{kj} &= pq^{-1}\partial_j\sigma_{ki}, & k < i < j \\
 \sigma_{ki}\partial_j &= pq^{-1}\sigma_{kj}\partial_i, & k < i < j \\
 \partial_i\sigma_{ij} - pq^{-1}\partial_j\sigma_{ii} &= pq^{-1}\sigma_{ij}\partial_i - \sigma_{ii}\partial_j, & i < j
 \end{aligned}$$

Proof. Let $\alpha, \beta \in \mathbb{N}^n$. To prove that the pair (∂, σ) is a right twisted multi-derivation, we show the following n equations hold

$$(6.43) \quad \partial_l(x^\alpha x^\beta) = \sum_k \partial_k(x^\alpha)\sigma_{kl}(x^\beta) + x^\alpha\partial_l(x^\beta), \quad l = 1, \dots, n.$$

Since $x_i x_j = q^{-1}x_j x_i$ for $i > j$, we have $x_i^{\alpha_i} x_j^{\beta_j} = q^{-\alpha_i \beta_j} x_j^{\beta_j} x_i^{\alpha_i}$ for $i > j$, and hence $x^\alpha x^\beta = \mu(\alpha, \beta)x^{\alpha+\beta}$, where $\mu(\alpha, \beta) = \prod_{1 \leq r < s \leq n} q^{-\alpha_s \beta_r}$. We then obtain

$$\partial_l(x^\alpha x^\beta) = \mu(\alpha, \beta)\delta_l(\alpha + \beta)x^{\alpha+\beta-\epsilon^l} = \pi_l(\alpha + \beta)\lambda_l(\alpha + \beta)\frac{p^{\alpha_l+\beta_l}-1}{p-1}\mu(\alpha, \beta)x^{\alpha+\beta-\epsilon^l}.$$

On the other hand, we compute

$$\begin{aligned}
\sum_{k=1}^n \partial_k(x^\alpha) \sigma_{kl}(x^\beta) &= \sum_{k=1}^{l-1} \partial_k(x^\alpha) \sigma_{kl}(x^\beta) + \partial_l(x^\alpha) \sigma_{ll}(x^\beta) \\
&= \sum_{k=1}^{l-1} \delta_k(\alpha) \pi_l(\beta) \bar{\lambda}_k(\beta) \lambda_l(\beta) (p^{\beta_l} - 1) x^{\alpha - \epsilon^k} x^{\beta + \epsilon^k - \epsilon^l} + p^{\beta_l} \delta_l(\alpha) \pi_l(\beta) \bar{\lambda}_l(\beta) \lambda_l(\beta) x^{\alpha - \epsilon^l} x^\beta \\
(6.44) \quad &= \left[\pi_l(\beta) \frac{p^{\beta_l} - 1}{p - 1} \sum_{k=1}^{l-1} \pi_k(\alpha) (p^{\alpha_k} - 1) + p^{\beta_l} \pi_l(\alpha + \beta) \frac{p^{\alpha_l} - 1}{p - 1} \right] \lambda_l(\alpha + \beta) \mu(\alpha, \beta) x^{\alpha + \beta - \epsilon^l} \\
&= \left[\pi_l(\beta) \frac{p^{\beta_l} - 1}{p - 1} (\pi_l(\alpha) - 1) + p^{\beta_l} \pi_l(\alpha + \beta) \frac{p^{\alpha_l} - 1}{p - 1} \right] \lambda_l(\alpha + \beta) \mu(\alpha, \beta) x^{\alpha + \beta - \epsilon^l} \\
&= \left[\pi_l(\alpha + \beta) \frac{p^{\alpha_l + \beta_l} - 1}{p - 1} - \pi_l(\beta) \frac{p^{\beta_l} - 1}{p - 1} \right] \lambda_l(\alpha + \beta) \mu(\alpha, \beta) x^{\alpha + \beta - \epsilon^l},
\end{aligned}$$

where the third equality holds because

$$\lambda_k(\alpha) \bar{\lambda}_k(\beta) x^{\alpha - \epsilon^k} x^{\beta + \epsilon^k - \epsilon^l} = \lambda_l(\alpha) \mu(\alpha, \beta) x^{\alpha + \beta - \epsilon^l} \quad \text{and} \quad x^{\alpha - \epsilon^l} x^\beta = \lambda_l(\beta) \mu(\alpha, \beta) x^{\alpha + \beta - \epsilon^l}.$$

The fourth equation follows since $\pi_k(\alpha) p^{\alpha_k} = \pi_{k+1}(\alpha)$. As we also have

$$(6.45) \quad x^\alpha \partial_l(x^\beta) = \delta_l(\beta) x^\alpha x^{\beta - \epsilon^l} = \pi_l(\beta) \lambda_l(\alpha + \beta) \frac{p^{\beta_l} - 1}{p - 1} \mu(\alpha, \beta) x^{\alpha + \beta - \epsilon^l}.$$

We can conclude, combining (6.44) and (6.45) that (6.43) holds:

$$(6.46) \quad \sum_{k=1}^n \partial_k(x^\alpha) \sigma_{kl}(x^\beta) + x^\alpha \partial_l(x^\beta) = \pi_l(\alpha + \beta) \lambda_l(\alpha + \beta) \frac{p^{\alpha_l + \beta_l} - 1}{p - 1} \mu(\alpha, \beta) x^{\alpha + \beta - \epsilon^l} = \partial_l(x^\alpha x^\beta).$$

For any $i < j$ we have:

$$(6.47) \quad \partial_i \partial_j(x^\alpha) = \delta_i(\alpha - \epsilon^j) \delta_j(\alpha) x^{\alpha - \epsilon^i - \epsilon^j} = q^{-1} \delta_i(\alpha) p \delta_j(\alpha - \epsilon^i) x^{\alpha - \epsilon^i - \epsilon^j} = pq^{-1} \partial_j \partial_i(x^\alpha)$$

For $i < k < j$, we have $\eta_{kj}(\alpha) = pq^{-1} \eta_{kj}(\alpha - \epsilon^i)$. Hence

$$(6.48) \quad \sigma_{kj} \partial_i(x^\alpha) = \delta_i(\alpha) \eta_{kj}(\alpha - \epsilon^i) x^{\alpha - \epsilon^i + \epsilon^k - \epsilon^j} = p^{-1} q \eta_{kj}(\alpha) \delta_i(\alpha) x^{\alpha - \epsilon^i + \epsilon^k - \epsilon^j} = p^{-1} q \partial_i(\sigma_{kj}(x^\alpha))$$

which shows that $\partial_i \sigma_{kj} = pq^{-1} \sigma_{kj} \partial_i$ for all $i < k < j$.

For $i < k = j$, we have $\eta_{jj}(\alpha) = pq^{-1} \eta_{jj}(\alpha - \epsilon^i)$. Thus

$$(6.49) \quad \partial_i \sigma_{jj}(x^\alpha) = \eta_{jj}(\alpha) \delta_i(\alpha) x^{\alpha - \epsilon^i} = pq^{-1} \delta_i(\alpha) \eta_{jj}(\alpha - \epsilon^i) x^{\alpha - \epsilon^i} = pq^{-1} \sigma_{jj}(\partial_i(x^\alpha)),$$

showing $\partial_i \sigma_{jj} = pq^{-1} \sigma_{jj} \partial_i$ for $i < j$.

For $k < i < j$ using $\eta_{kj}(\alpha) \delta_i(\alpha) = \eta_{ki}(\alpha) \delta_j(\alpha)$ we get:

$$\begin{aligned}
(6.50) \quad \partial_i \sigma_{kj}(x^\alpha) &= \eta_{kj}(\alpha) \delta_i(\alpha + \epsilon^k - \epsilon^j) x^{\alpha - \epsilon^i + \epsilon^k - \epsilon^j} \\
&= pq^{-1} \eta_{kj}(\alpha) \delta_i(\alpha) x^{\alpha - \epsilon^i + \epsilon^k - \epsilon^j} \\
&= pq^{-1} \eta_{ki}(\alpha) \delta_j(\alpha) x^{\alpha - \epsilon^i + \epsilon^k - \epsilon^j} \\
&= pq^{-1} \eta_{ki}(\alpha) \delta_j(\alpha + \epsilon^k - \epsilon^i) x^{\alpha - \epsilon^i + \epsilon^k - \epsilon^j} = pq^{-1} \partial_j \sigma_{ki}(x^\alpha)
\end{aligned}$$

showing $\partial_i \sigma_{kj}(x^\alpha) - pq^{-1} \partial_j \sigma_{ki}(x^\alpha) = 0$. In a similar way, the relation

$$pq^{-1} \sigma_{kj} \partial_i(x^\alpha) - \sigma_{ki} \partial_j(x^\alpha) = 0$$

holds for $k < i < j$. Lastly, we show that the equations

$$\partial_i \sigma_{ij}(x^\alpha) - pq^{-1} \partial_j \sigma_{ii}(x^\alpha) = pq^{-1} \sigma_{ij} \partial_i(x^\alpha) - \sigma_{ii} \partial_j(x^\alpha), \quad i < j$$

are satisfied, because of the following equations for $i < j$

$$\sigma_{ii} \partial_j(x^\alpha) = \frac{q^{-1} p^{\alpha_i}}{p - 1} \eta_{ij}(\alpha) \pi_i(\alpha) \lambda_i(\alpha) x^{\alpha - \epsilon^j} = q^{-1} \partial_j \sigma_{ii}(x^\alpha)$$

$$\partial_i \sigma_{ij}(x^\alpha) = \frac{q^{-1}}{p-1} \eta_{ij}(\alpha) \pi_i(\alpha) \lambda_i(\alpha) (p^{\alpha_i+1} - 1) x^{\alpha-\epsilon^j},$$

$$\sigma_{ij} \partial_i(x^\alpha) = \frac{p^{-1}(p^{\alpha_i} - 1)}{p-1} \eta_{ij}(\alpha) \pi_i(\alpha) \lambda_i(\alpha) x^{\alpha-\epsilon^j},$$

By using these equations we attain the equation:

$$\partial_i \sigma_{ij}(x^\alpha) - pq^{-1} \partial_j \sigma_{ii}(x^\alpha) = -\frac{q^{-1}}{p-1} \eta_{ij}(\alpha) \pi_i(\alpha) \lambda_i(\alpha) x^{\alpha-\epsilon^j}$$

and

$$pq^{-1} \sigma_{ij} \partial_i(x^\alpha) - \sigma_{ii} \partial_j(x^\alpha) = -\frac{q^{-1}}{p-1} \eta_{ij}(\alpha) \pi_i(\alpha) \lambda_i(\alpha) x^{\alpha-\epsilon^j},$$

which completes the proof the lemma. \square

Denote by $\Omega = \bigwedge^{p^{-1}q}(\Omega^1)$ the quantum exterior algebra of Ω^1 over $K_q[x_1, \dots, x_n]$ with respect to the matrix Q' .

Theorem 6.3. *The derivation $d : K_q[x_1, \dots, x_n] \rightarrow \Omega^1$ extends to a differential calculus $\bigwedge^{p^{-1}q}(\Omega^1)$ on $K_q[x_1, \dots, x_n]$. Furthermore the de Rham and the integral complex associated to the differential calculus $(\bigwedge^{p^{-1}q}(\Omega^1), d)$ are isomorphic.*

Proof. The first statement follows from Proposition 3.1 and Lemma 6.2. We have an upper-triangular $\sigma = (\sigma_{ij})$ matrix by Lemma 6.1, of which the diagonal entries $\sigma_{ii}, i = 1, \dots, n$ are automorphisms. Hence we construct the corresponding lower-triangular matrix $\bar{\sigma}$ according to [7, Proposition 3.3]. The entries of $\bar{\sigma}$ are $\bar{\sigma}_{ij} = 0$ for $i < j$ and $\bar{\sigma}_{ii} = \sigma_{ii}^{-1}$ while

$$(6.51) \quad \bar{\sigma}_{ij}(x^\alpha) = q \pi_i(\alpha)^{-1} \bar{\lambda}_j(\alpha)^{-1} \lambda_i(\alpha)^{-1} (p^{-\alpha_i} - 1) q^{\alpha_j - \alpha_i} x^{\alpha + \epsilon^j - \epsilon^i},$$

for $\alpha \in \mathbb{N}^n$ and $i > j$. Applying [7, Proposition 3.3] again yields the map $\hat{\sigma}$. The entries of $\hat{\sigma}$ are $\hat{\sigma}_{ij} = 0$ for $i > j$ and $\hat{\sigma}_{ii} = \sigma_{ii}$ while $\hat{\sigma}_{ij} = p^{j-i} \sigma_{ij}$ for $i < j$.

By using these formulas for the entries of the matrices $\bar{\sigma}(x^\alpha)$ and $\hat{\sigma}(x^\alpha)$, we obtain an explicit expression for

$$\partial_i^\sigma(x^\alpha) = \sum_{1 \leq j \leq k \leq i} \bar{\sigma}_{kj} \circ \partial_j \circ \hat{\sigma}_{ki}(x^\alpha).$$

for any fixed $i = 1, \dots, n$. For $j < k < i$ we get:

$$\bar{\sigma}_{kj} \circ \partial_j \circ \hat{\sigma}_{ki}(x^\alpha) = -p^{i-k} \pi_j(\alpha) \pi_k(\alpha)^{-1} (p - p^{-\alpha_k}) (p^{\alpha_j} - 1) \partial_i(x^\alpha)$$

while for $j = k < i$ we have:

$$\bar{\sigma}_{kk} \circ \partial_k \circ \hat{\sigma}_{ki}(x^\alpha) = p^{i-k} (p - p^{-\alpha_k}) \partial_i(x^\alpha)$$

Thus for any $k < i$ we get the partial sum:

$$\begin{aligned} \Lambda_k &= \sum_{j=1}^k \bar{\sigma}_{kj} \circ \partial_j \circ \hat{\sigma}_{ki}(x^\alpha) \\ &= \sum_{j=1}^{k-1} -p^{i-k} \pi_j(\alpha) \pi_k(\alpha)^{-1} (p - p^{-\alpha_k}) (p^{\alpha_j} - 1) \partial_i(x^\alpha) + p^{i-k} (p - p^{-\alpha_k}) \partial_i(x^\alpha) \\ &= \left[1 - \sum_{j=1}^{k-1} \pi_j(\alpha) (p^{\alpha_j} - 1) \pi_k(\alpha)^{-1} \right] p^{i-k} (p - p^{-\alpha_k}) \partial_i(x^\alpha) \\ &= [\pi_k(\alpha) - \pi_k(\alpha) + 1] \pi_k(\alpha)^{-1} p^{i-k} (p - p^{-\alpha_k}) \partial_i(x^\alpha) = \pi_k(\alpha)^{-1} p^{i-k} (p - p^{-\alpha_k}) \partial_i(x^\alpha) \end{aligned}$$

Similarly, for $k = i$ we have for $j < k = i$: $\bar{\sigma}_{ij} \circ \partial_j \circ \hat{\sigma}_{ji}(x^\alpha) = -p\pi_j(\alpha)(p^{\alpha_j} - 1)\pi_i(\alpha)^{-1}\partial_i(x^\alpha)$ and for $j = k = i$ we have $\bar{\sigma}_{ii} \circ \partial_i \circ \hat{\sigma}_{ii}(x^\alpha) = p\partial_i(x^\alpha)$. This gives

$$\Lambda_i = \sum_{j=1}^i \bar{\sigma}_{ij} \circ \partial_j \circ \hat{\sigma}_{ii}(x^\alpha) = p\pi_i(\alpha)^{-1}\partial_i(x^\alpha).$$

The sum of these partial sums Λ_k yields:

$$\begin{aligned} \partial_i^\sigma(x^\alpha) &= \sum_{k=1}^i \Lambda_k = \sum_{k=1}^{i-1} \pi_k(\alpha)^{-1} p^{i-k} (p - p^{-\alpha_k}) \partial_i(x^\alpha) + p\pi_i(\alpha)^{-1} \partial_i(x^\alpha) \\ &= \left[\sum_{k=1}^{i-1} \pi_k(\alpha)^{-1} p^{i-k} (p - p^{-\alpha_k}) + p\pi_i(\alpha)^{-1} \right] \partial_i(x^\alpha) \\ &= p\lambda_i(\alpha) \frac{p^{\alpha_i} - 1}{p - 1} \left[1 + p^{i-1} \sum_{k=1}^{i-1} p^{-k} (p^{\alpha_k+1} - 1) \left(\prod_{k < s < i} p^{\alpha_s} \right) \right] x^{\alpha - \epsilon^i} \\ &= p\lambda_i(\alpha) \frac{p^{\alpha_i} - 1}{p - 1} \left[1 + p^{i-1} \sum_{k=1}^{i-1} \left((p^{-(k-1)} \left(\prod_{k-1 < s < i} p^{\alpha_s} \right) - p^{-k} \left(\prod_{k < s < i} p^{\alpha_s} \right) \right) \right] x^{\alpha - \epsilon^i} \\ &= p\lambda_i(\alpha) \frac{p^{\alpha_i} - 1}{p - 1} \left[1 + p^{i-1} (\pi_i(\alpha) - p^{-(i-1)}) \right] x^{\alpha - \epsilon^i} \\ &= p^i \partial_i(x^\alpha) \end{aligned}$$

In order to apply Theorem 3.2, we need to calculate $\det(\sigma)$ as well as $\prod_j q'_{ij}$ where $Q' = (q'_{ij})$ is the corresponding multiplicatively antisymmetric matrix with $q'_{ij} = p^{-1}q$ for $i < j$. Let $\alpha \in \mathbb{N}^n$. By Theorem 3.2 it is enough to show that $\partial_i^\sigma(x^\alpha) = \left(\prod_j q'_{ij} \right) \det(\sigma)^{-1} (\partial_i(\det(\sigma)(x^\alpha)))$ holds, i.e.

$$p^i \partial_i(x^\alpha) = \left(\prod_j q'_{ij} \eta_{jj}(\alpha) \eta_{jj}(\alpha - \epsilon^i)^{-1} \right) \partial_i(x^\alpha).$$

By the definition of η_{ij} we obtain $p^{-1}q\eta_{jj}(\alpha)\eta_{jj}(\alpha - \epsilon^i)^{-1} = 1$ for $i < j$ and $pq^{-1}\eta_{jj}(\alpha)\eta_{jj}(\alpha - \epsilon^i)^{-1} = p$ for $i > j$, while $\eta_{ii}(\alpha)\eta_{ii}(\alpha - \epsilon^i)^{-1} = p$. Hence the product of the $q'_{ij}\eta_{jj}(\alpha)\eta_{jj}(\alpha - \epsilon^i)^{-1}$ equals p^i and by Theorem 3.2 $K_q[x_1, \dots, x_n]$ satisfies the strong Poincaré duality with respect to the differential calculus $(\bigwedge^{p^{-1}q}(\Omega^1), d)$. □

7. CONCLUSION

Necessary and sufficient conditions to extend the associated FODC (Ω^1, d) of a right twisted multi-derivation (∂, σ) on an algebra A to a full differential calculus (Ω, d) on the quantum exterior algebra Ω of Ω^1 have been presented in this paper. A chain map between the de Rham complex and the integral complex has been defined and an criterion has been given to assure an isomorphism between the de Rham and the integral complexes for free right upper-triangular twisted multi-derivations whose associated FODC can be extended to a full differential calculus on the quantum exterior algebra. Easier criteria for FODCs with a diagonal bimodule structure have been established and have been applied to show that a multivariate quantum polynomial algebra satisfies the strong Poincaré duality in the sense of T.Brzeziński with respect to some canonical FODC. Lastly, we showed that for a certain two-parameter n -dimensional (upper-triangular) calculus over Manin's quantum n -space the de Rham and integral complexes are isomorphic.

Future work will consist in extending our duality criteria to general FODCs having an upper-triangular bimodule structure.

REFERENCES

- [1] V. A. Artamonov, *Quantum polynomials.* in Shum, K. P. (ed.) et al., *Advances in algebra and combinatorics. Proceedings of the 2nd international congress in algebra and combinatorics*, Guangzhou, China, 2007. Hackensack, NJ: World Scientific 19-34 (2008).
- [2] V. A. Artamonov and R. Wisbauer, *Homological properties of quantum polynomials.*, Algebr. Represent. Theory 4(3) (2001), 219-247
- [3] K. A. Brown and K. R. Goodearl, *Lectures on Algebraic Quantum Groups*, Advanced courses in mathematics, CRM Barcelona, Birkhäuser, Basel, Switzerland, 2002
- [4] T. Brzeziński, *Non-commutative connections of the second kind*, J. Algebra Appl. 7 (2008), 557–573
- [5] T. Brzeziński, *Integral calculus on $E_q(2)$* , SIGMA Symmetry Integrability Geom. Methods Appl. 6 (2010), Paper 040, 10 pp.
- [6] T. Brzeziński, *Divergences on projective modules and non-commutative integrals.*, Int. J. Geom. Methods Mod. Phys. 8(4) (2011), 885-896
- [7] T. Brzeziński, L. El Kaoutit, C. Lomp, *Non-commutative integral forms and twisted multi-derivations*, J. Noncommut. Geom. 4 (2010), 281–312
- [8] A. Connes, *Non-commutative differential geometry*, Inst. Hautes Études Sci. Publ. Math. 62 (1985), 257–360
- [9] A. Klimyk and K. Schmüdgen, *Quantum Groups and Their Representations*, Springer, Berlin, Germany, 1997
- [10] U. Krähmer, *Poincaré duality in Hochschild (co)homology.* in New techniques in Hopf algebras and graded ring theory, 117–125, K. Vlaam. Acad. Belgie Wet. Kunsten (KVAB), Brussels, 2007.
- [11] D. Naidu, P. Shroff, S. Whitherspoon, *Hochschild cohomology of group extensions of quantum symmetric algebras*, Proc. Amer. Math. Soc. 139 (2011), 1553-1567
- [12] K. Ueyama, *Graded Frobenius algebras and quantum Beilinson algebras.* in *Proceedings of the 44th Symposium on Ring Theory and Representation Theory*, 216–222, Symp. Ring Theory Represent. Theory Organ. Comm., Nagoya, 2012
- [13] M. van den Bergh, *A relation between Hochschild homology and cohomology for Gorenstein rings.*, Proc. Amer. Math. Soc. 126 (1998), no. 5, 1345–1348.
- [14] S.L. Woronowicz, *Twisted SU(2) group. An example of a noncommutative differential calculus.* Publ. Res. Inst. Math. Sci. 23 (1987), 117–181

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